2/3 UNIT MATHEMATICS FORM VI

Time allowed: 3 hours (plus 5 minutes reading) **Exam date:** 11th August, 1998

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the left margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

QUESTION ONE (Start a new answer booklet)

Marks

2 (a) Solve the equation $\frac{2t}{5} + 14 = 8$.

2 (b) If $m_1 = 34, m_2 = 7, M = 53$ and $g = 9{\cdot}8$ find, correct to 4 significant figures, the value of

$$\left(\frac{m_1 - m_2}{M + m_1 + m_2}\right) g .$$

2 (c) The line $kx - 2y = 23$ passes through the point $(3, -1)$. Find the value of k.

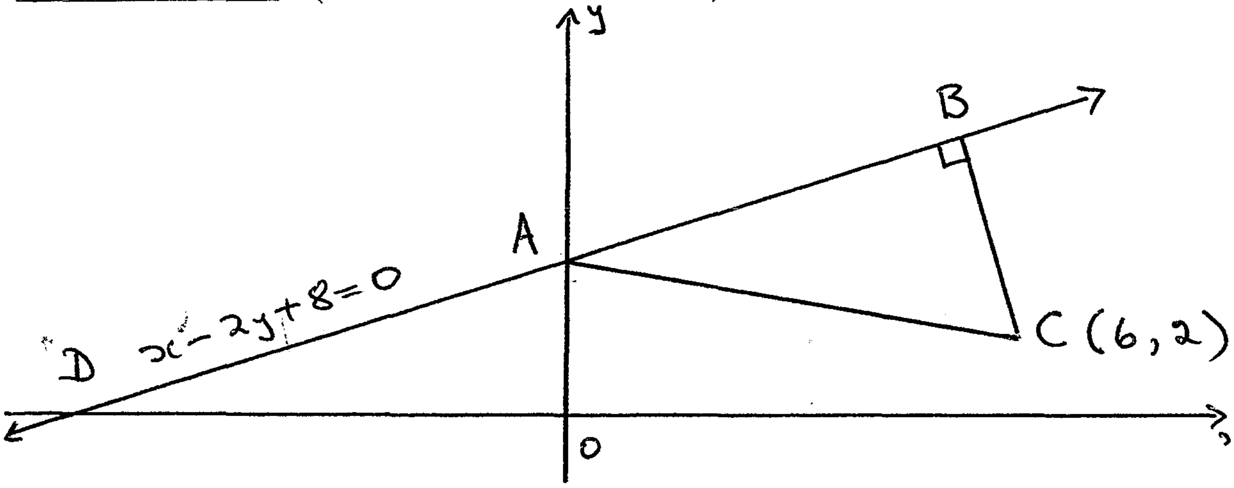
2 (d) Simplify $\frac{x}{4} + \frac{3x-1}{3}$.

2 (e) Factorise $3x^2 + 5x - 12$.

2 (f) Solve $7 - 4x > 12$.

Exam continues next page ...

QUESTION TWO (Start a new answer booklet)



In the diagram above, the line $x - 2y + 8 = 0$ cuts the x-axis at D and the y-axis at A. From the point $C(6,2)$, a perpendicular is drawn to the line, meeting it at B.

Marks

[1] (a) Show that A is the point $(0,4)$.

[2] (b) Find the gradients m_1 of AC and m_2 of BC.

[2] (c) Show that the equation of BC is $2x + y - 14 = 0$.

[2] (d) Find the coordinates of B.

[2] (e) Find the lengths of AB and BC.

[2] (f) Find the area of triangle ABC.

[1] (g) Find, to the nearest degree, the size of angle ADO.

QUESTION THREE (Start a new answer booklet)

Marks

6 (a) Differentiate with respect to x:

(i) $(x^2-3)^3$,

(ii) $\log_e(x^2+3)$,

(iii) $x\cos x$.

2 (b) The graph of $y=f(x)$ passes through the point $(3,5)$, and $f'(x)=2x-3$. Find $f(x)$.

4 (c) Find:

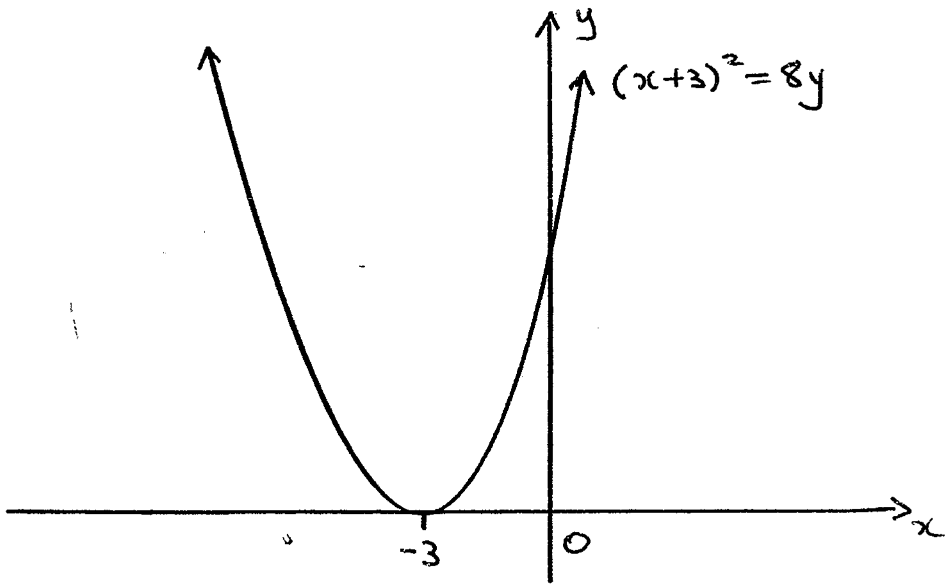
(i) $\int \sqrt{x+6}\,dx$,

(ii) $\int_0^{\frac{\pi}{8}} \sec^2 2x\,dx$.

QUESTION FOUR (Start a new answer booklet)

Marks

[4] (a)



The diagram is of the parabola $(x+3)^2 = 8y$.

(i) Find the coordinates of the vertex of the parabola.

(ii) Find the focal length of the parabola.

(iii) Find the coordinates of the focus of the parabola.

(iv) Find the equation of the directrix of the parabola.

[3] (b) The table shows the value of a function $f(x)$ for five values of x.

x	0	2	4	6	8
$f(x)$	4	7	1	-3	-1

Use Simpson's rule with these five values to estimate $\int_0^8 f(x)\,dx$.

[5] (c) (i) Sketch the graph of $y = 2\sin x$ for $0 \le x \le 2\pi$.

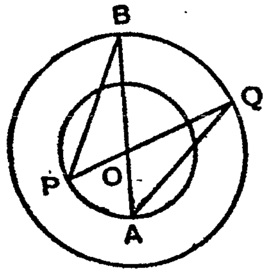
(ii) On the same set of axes, sketch the graph of $y = 2\sin x + 1$ for $0 \le x \le 2\pi$.

(iii) Find the exact values of the x coordinates of the points where the graph of $y = 2\sin x + 1$ crosses the x axis in the domain $0 \le x \le 2\pi$.

QUESTION FIVE (Start a new answer booklet)

Marks

[4] (a)



In the diagram, O is the centre of each of the circles. AOB and POQ are straight lines.

(i) Prove that $\triangle OAQ \equiv \triangle OPB$.

(ii) Hence prove that $AQ = PB$

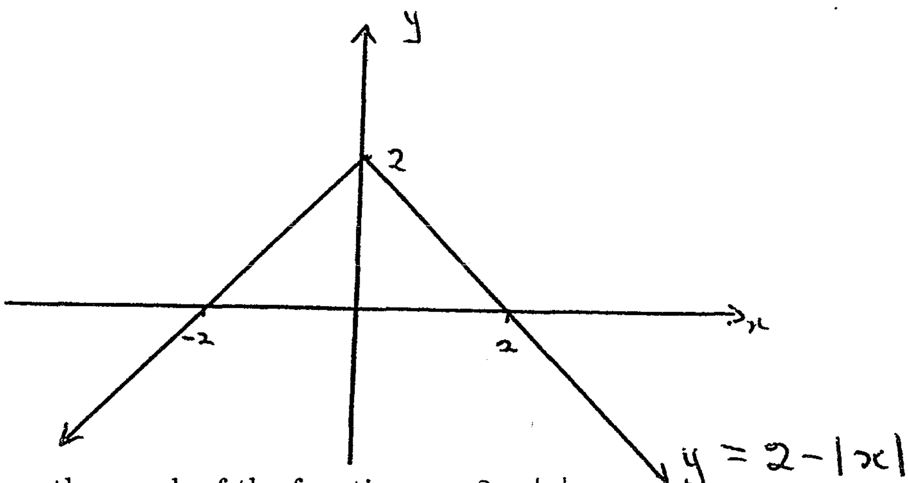
[5] (b) A length of railway track is to be relaid and supplies of materials are to be deposited at 24 points along the track at $\frac{1}{4}$ km intervals. Supplies are taken by a train from a depot which is 15 km from the nearest of these points. The train must return to the depot and reload after depositing supplies at each individual point.

(i) How far has the train travelled when it returns to the depot after it has deposited the materials at the point nearest to the depot?

(ii) How far is the farthest point from the depot?

(iii) Use the sum of an arithmetic progression to find the total distance the train must travel in order to return to the depot having supplied all the points with materials?

[3] (c)



The diagram shows the graph of the function $y = 2 - |x|$.

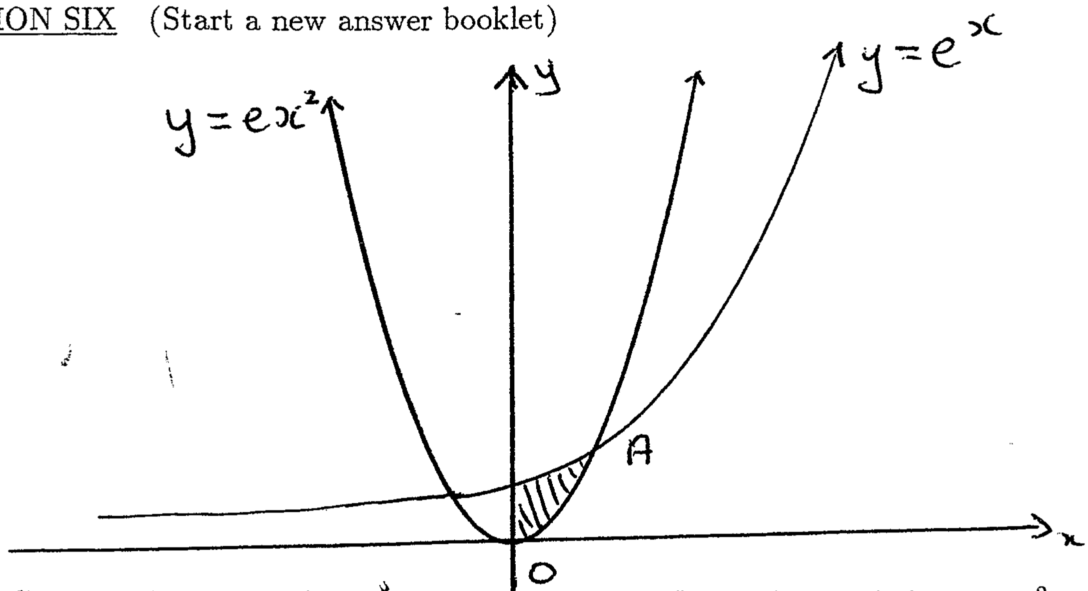
(i) Copy the diagram into your answer booklet, and draw on your diagram the line $x + 2y = 0$.

(ii) Find the coordinates of the points where the line intersects $y = 2 - |x|$.

QUESTION SIX (Start a new answer booklet)

Marks

[4] (a)

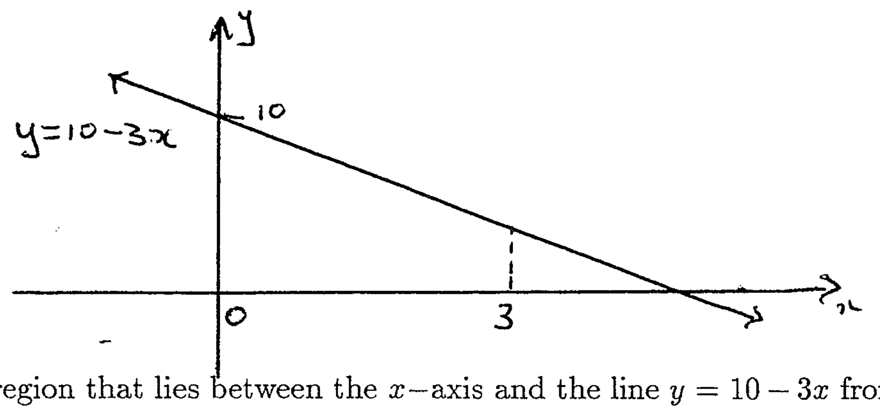


The diagram above is of the exponential curve $y = e^x$ and the parabola $y = ex^2$. The point A is the first point where the two graphs meet on the right-hand side of the Y axis.

(i) Show that A is the point $(1, e)$.

(ii) Show that the shaded area is $\frac{2e}{3} - 1$ units^2.

[3] (b)



The region that lies between the x−axis and the line $y = 10 - 3x$ from $x = 0$ to $x = 3$ is rotated about the x−axis to form a solid. Find the exact volume of the solid.

[5] (c) The position x cm at time t seconds of a particle moving in a straight line is given by $x = 3t + e^{-3t}$.

(i) Find the position of the particle when $t = 1$. (Give your answer correct to three significant figures).

(ii) By finding an expression for the velocity of the particle, show that initially the particle is at rest.

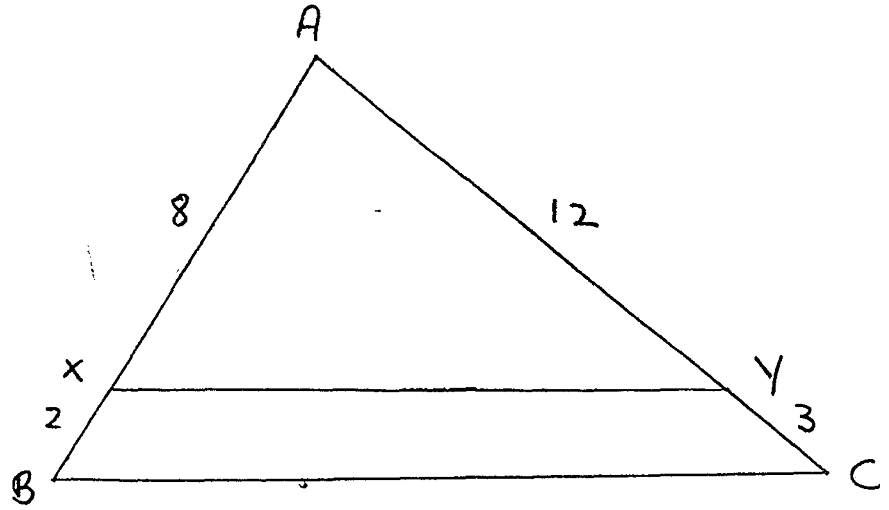
(iii) Find an expression for the acceleration of the particle.

(iv) Find the limiting velocity of the particle as $t \to \infty$.

QUESTION SEVEN (Start a new answer booklet)

Marks

4 (a)

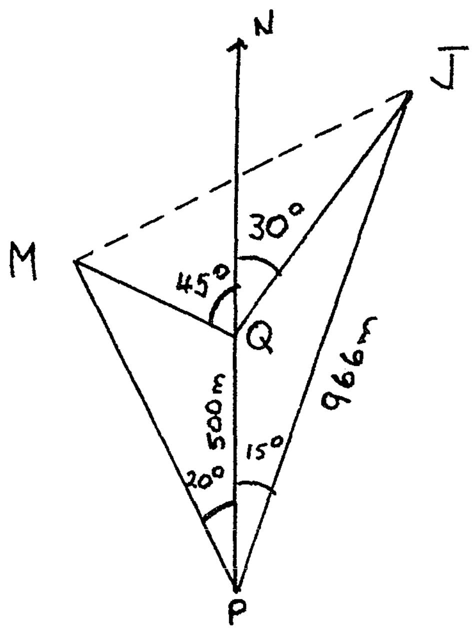


In the diagram above, AB and AC are straight lines. $AX = 8, XB = 2, AY = 12$ and $YC = 3$.

(i) Prove that $\triangle ABC \parallel\!| \triangle AXY$.

(ii) Prove that XY is parallel to BC.

6 (b)



A walker at point P can see the spires of St Michael's (M) and St John's (J) on bearings of 340° and 015° respectively. She then walks 500 metres due north to a point Q and notes the bearings as 315° to St Michael's and 030° to St John's. The distance from P to St John's is known to be 966 metres. All the measurements are taken in the same horizontal plane.

(i) Copy the diagram into your answer booklet and fill in the information given.

(ii) Show that the distance PM is 837 m to the nearest metre.

(iii) Find, correct to the nearest metre, the distance between the two spires.

2 (c) By expressing $\cot\theta$ and $\operatorname{cosec}\theta$ in terms of $\sin\theta$ and $\cos\theta$, show that:

$$\operatorname{cosec}^2\theta - \cot^2\theta = 1\,.$$

QUESTION EIGHT (Start a new answer booklet)

Marks

5 (a) A substance is known to decompose at a rate proportional to the mass present at any time. That is, $\dfrac{dm}{dt} = km$, where m is the mass of the substance present after t weeks and k is a constant.

(i) Show that if A is a constant then $m = Ae^{kt}$ is a solution of the equation $\dfrac{dm}{dt} = km$.

(ii) It is known that 700 grams of the substance described above decomposes to 600g in 11 weeks.

(α) Show that $A = 700$.

(β) Show that $k = \frac{1}{11}\log_e \frac{6}{7}$.

(γ) To the nearest gram, how much of the substance remains after 100 weeks?

(δ) After how many weeks will there be less than 100 g of the substance remaining?

7 (b) A pendulum is set swinging by lifting it to the right and releasing it. Its first swing (from right to left) is through an angle of $40°$. It's next swing (from left to right) is through $36°$, and each succeeding swing is through 90% of the angle of the swing before.

(i) List the first four angles (in degrees correct to two decimal places where necessary) through which the pendulum swings.

(ii) Explain why these four values form a geometric sequence.

(iii) Find an expression for the nth term of this sequence.

(iv) Which swing will be the first swing through less than $1°$?

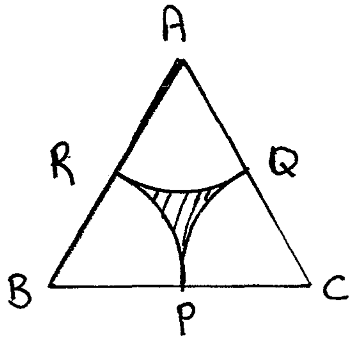
(v) What is the sum of all the angles it has swung through at the end of its 10th oscillation? Give your answer to the nearest degree.

(vi) Explain why it is impossible for the total angle the pendulum swings through to be greater than $400°$.

QUESTION NINE (Start a new answer booklet)

Marks

3 (a)



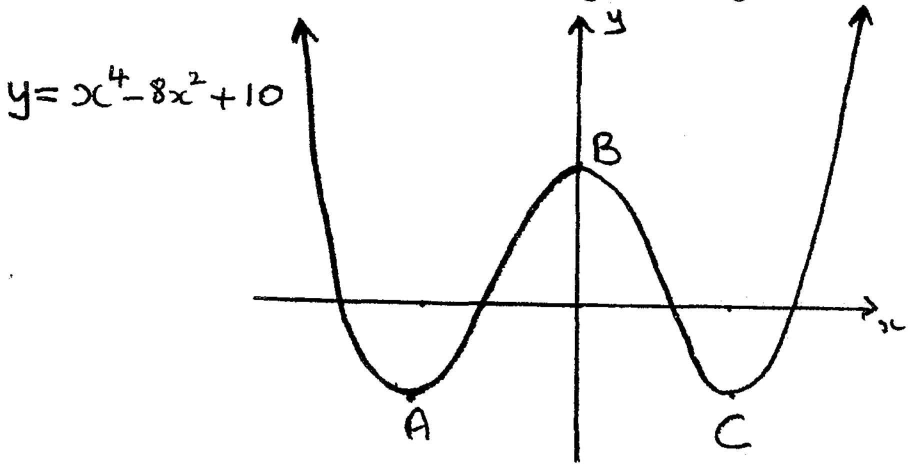
The triangle ABC in the diagram above is equilateral and each side is 12 cm long. P, Q and R are the mid-points of BC, AC and AB respectively. RP, PQ and QR are arcs of circles centred B, C and A respectively.

(i) Show that the area of triangle ABC is $36\sqrt{3}\,\text{cm}^2$.

(ii) Find the exact area of the sector ARQ.

(iii) Hence find the shaded area correct to three significant figures.

6 (b)



The graph above is of the function $y = f(x)$, where $f(x) = x^4 - 8x^2 + 10$. A and C are minimum turning points and B is the maximum turning point where the graph cuts the y-axis.

(i) Find the co-ordinates of B.

(ii) Find $f'(x)$.

(iii) Show that the solutions of the equation $f'(x) = 0$ are $x = 0$, $x = -2$ and $x = 2$.

(iv) Hence find the coordinates of A and C, confirm that they are minima and that B is a maximum.

3 (c) If $0 \le x \le 2\pi$, find all solutions of $2\cos^2(\frac{\pi}{3} - x) = 1$.

QUESTION TEN (Start a new answer booklet)

Marks

4 (a) When a ship is travelling at a speed of v kilometres per hour, its rate of consumption of fuel in tonnes per hour is approximately $50 + 0{\cdot}002v^3$.

(i) Show that on a voyage of 9000 km at a speed v km/h the expression for the total fuel used, A tonnes, is given by:

$$A = \frac{450\ 000}{v} + 18v^2 .$$

(ii) Hence find the speed for the greatest fuel economy and the amount of fuel used at this speed. Give both your answers to three significant figures.

8 (b) Consider the two functions $f(x)$ and $g(x)$ where $f(x) = \dfrac{e^x - e^{-x}}{2}$ and $g(x) = \dfrac{e^x + e^{-x}}{2}$.

(i) Show that $f'(x) = g(x)$ and $g'(x) = f(x)$.

(ii) Show that the graph $y = f(x)$ is increasing for all values of x and that there is a point of inflexion at $x = 0$.

(iii) Show that the graph of $y = g(x)$ has a minimum at $x = 0$.

(iv) Sketch the graphs of $y = f(x)$ and $y = g(x)$.

(v) Let $y = f(x)$. Show that this equation can be written in the form:

$$e^{2x} - 2ye^x - 1 = 0 .$$

Hence deduce that $x = \log_e(y + \sqrt{y^2 + 1})$.

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The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

$$\text{NOTE: } \ln x = \log_e x, \quad x > 0$$

EACH QUESTION IS OUT OF 12
$\Longrightarrow$ TOTAL MARKS = 120

QUESTION ONE

(a) $\dfrac{2t}{5} + 14 = 8$
$\Longrightarrow 2t = -30$
$\Longrightarrow t = -15$ [2]

(b) $\left(\dfrac{34-7}{53+34+7}\right) \times 9{\cdot}8 = 2{\cdot}8148936\ldots$
$= 2{\cdot}815$ (4 sig-figs) [2]

(c) $3k + 2 = 23$
$\Longrightarrow 3k = 21$
$\Longrightarrow k = 7$ [2]

(d) $\dfrac{x}{4} + \dfrac{3x-1}{3}$
$= \dfrac{3x + 12x - 4}{12}$
$= \dfrac{15x - 4}{12}$ [2]

(e) $3x^2 + 5x - 12 = (3x-4)(x+3)$ [2]

(f) $7 - 4x > 12$
$\Longrightarrow -4x > 5$
$\Longrightarrow x < -\frac{5}{4}$ [2]

QUESTION TWO

(a) At A, $x = 0$.
i.e. $-2y + 8 = 0$
$\Longrightarrow y = 4$
$\Longrightarrow$ A is the point $(0, 4)$. [1]

(b) $m_1 = \dfrac{4-2}{-6} = -\dfrac{1}{3}$ [1]
As $BC \perp DAB$, $m_2 = -2$ [1]

(c) BC is: $\dfrac{y-2}{x-6} = -2$
$\implies y - 2 = -2x + 12$
i.e. $2x + y - 14 = 0$ [2]

(d) $2x + y = 14 \ldots 1$
$x - 2y = -8 \ldots 2$
$4x + 2y = 28 \ldots 1 \times 2$
$\implies 5x = 20 \ldots 2 + 1 \times 2$
$\implies x = 4 \implies y = 6$
i.e. B is $(4, 6)$ [2]

(e) $AB = \sqrt{4^2 + (6-4)^2}$
$\implies AB = 2\sqrt{5}$ [1]
$BC = \sqrt{(6-4)^2 + (2-6)^2}$
$\implies BC = 2\sqrt{5}$ [1]

(f) Area $= \frac{1}{2}(2\sqrt{5})^2 = 10\,\text{units}^2$ [2]

(g) $\tan \angle ADO = \frac{1}{2} \implies \angle ADO = 27^\circ$, to the nearest degree. [1]

QUESTION THREE

(a) (i) $6x(x^2 - 3)^2$ [2]

(ii) $\dfrac{2x}{x^2 + 3}$ [2]

(iii) $\cos x - x \sin x$ [2]

(b) $f'(x) = 2x - 3$
$\implies f(x) = x^2 - 3x + c,\ c \in \mathbf{R}$
But $f(3) = 5 \implies 9 - 9 + c = 5$
So $c = 5$
$\implies f(x) = x^2 - 3x + 5$ [2]

(c) (i) $\displaystyle\int \sqrt{x+6}\,dx = \frac{2}{3}(x+6)^{\frac{3}{2}} + c,\ c \in \mathbf{R}$ [2]
Note: Constant of integration not necessary for full marks.

(ii) $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2 2x\,dx$
$= \left[\frac{1}{2}\tan 2x\right]_0^{\frac{\pi}{8}}$
$= \frac{1}{2}\left[\tan\frac{\pi}{4} - \tan 0\right]$
$= \frac{1}{2}$ [2]

QUESTION FOUR

(a) (i) Vertex is $(-3, 0)$ [1]

(ii) Focal length $= 2$ [1]

(iii) Focus is $(-3, 2)$ [1]

(iv) The directrix is the line $y = -2$ [1]

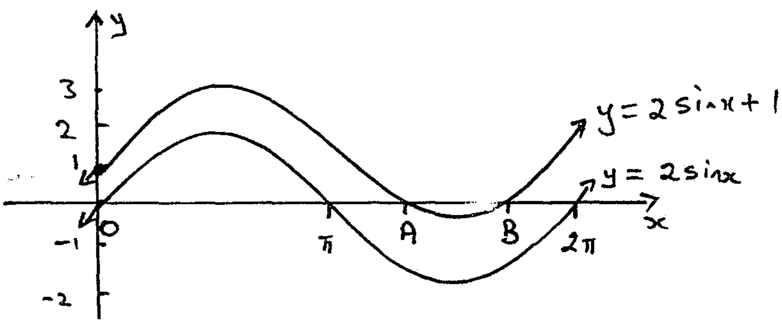
(b) $h = 2$

So $\int_0^8 f(x)\,dx \doteqdot \frac{2}{3}\left[4 + 4(7-3) + 2 \times 1 - 1\right]$

$\doteqdot \frac{2}{3}\left[21\right]$

So $\int_0^8 f(x)\,dx \doteqdot 14$ [3]

(c) (i)



[2]

(ii) See above [1]

(iii) At A : $\sin x = -\frac{1}{2} \Longrightarrow x = \frac{7\pi}{6}$

At B : $\sin x = -\frac{1}{2} \Longrightarrow x = \frac{11\pi}{6}$ [2]

QUESTION FIVE

(a) (i) In $\triangle$'s OAQ and OPB

(α) $OA = OP$ (radii)

(β) $OQ = OB$ (radii)

(γ) $\angle AOQ = \angle POB$ (vert. opp.)
Hence $\triangle OAQ \equiv \triangle OPB$ (SAS) [3]

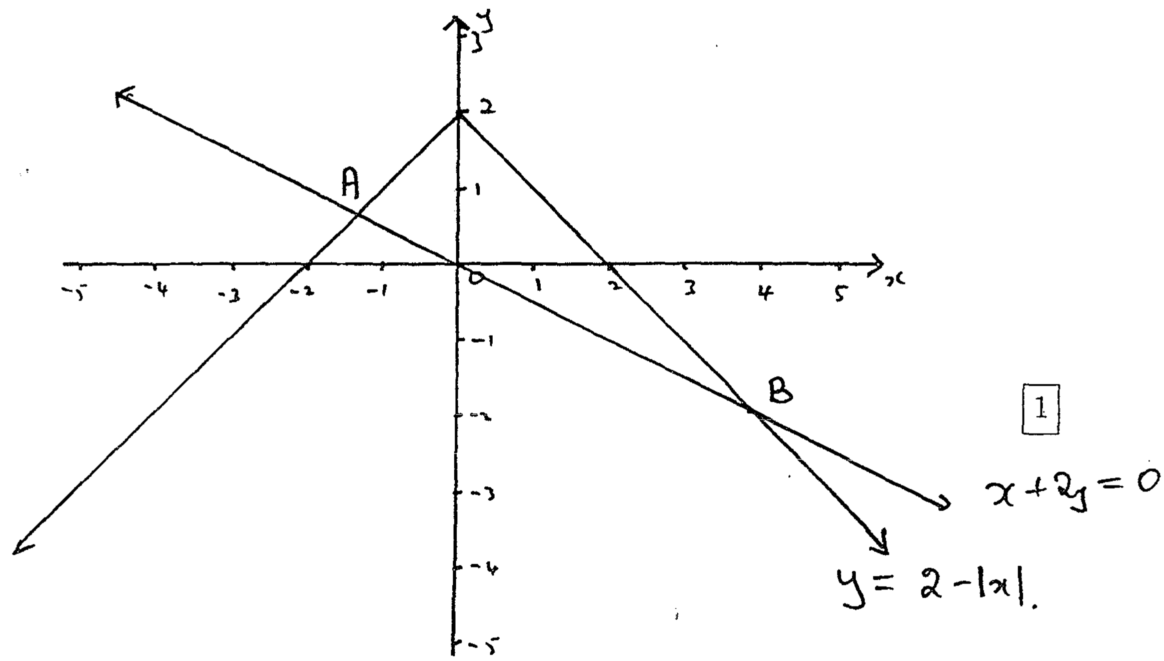
(ii) Hence $AQ = PB$ (matching sides of $\equiv$ $\triangle$'s equal) [1]

(b) (i) 30 km [1]

(ii) Distance of farthest $= u_{24} = 15 + \frac{1}{4} \times 23 = 20{\cdot}75$ km [2]

(iii) Total distance is given by S_{24}
$S_{24} = 2(15 + 15{\cdot}25 + 15{\cdot}50 + \ldots + 20{\cdot}75)$ or $= 2 \times \frac{24}{2}(u_1 + u_{24})$
$S_{24} = 2 \times 12(30 + 23 \times 0{\cdot}25)$ or $= 24(15 + 20{\cdot}75)$
$S_{24} = 858$ km [2]

(c) (i)



[1]

(ii) At A : $x + 2y = 0$ and $y = 2 + x$
i.e. $x + 4 + 2x = 0$
$\Longrightarrow x = -\frac{4}{3} \Longrightarrow A$ is $(-\frac{4}{3}, \frac{2}{3})$
At B : $x + 2y = 0$ and $y = 2 - x$
i.e. $x + 4 - 2x = 0$
$\Longrightarrow x = 4 \Longrightarrow B$ is $(4, -2)$ [2]

QUESTION SIX

(a) (i) At A $ex^2 = e^x$
When $x = 1$, LHS $= e$, RHS $= e$
So A is $(1, e)$ [1]

(ii) Area $= \int_0^1 e^x - ex^2\,dx$
$\Longrightarrow A = \left[e^x - \frac{e}{3}x^3\right]_0^1$
$\Longrightarrow A = e - \frac{e}{3} - 1 + 0$
$\Longrightarrow A = \frac{2e}{3} - 1$ units^2 as required. [3]

(b) $V = \pi \int_0^3 (10 - 3x)^2\,dx$
$\Longrightarrow V = \pi \int_0^3 100 - 60x + 9x^2\,dx$
$\Longrightarrow V = \pi \left[100x - 30x^2 + 3x^3\right]_0^3$ or $-\frac{\pi}{9}\left[(10 - 3x)^3\right]_0^3$
$\Longrightarrow V = \left[300 - 270 + 81 - 0\right]$ or $-\frac{\pi}{9}\left[1^3 - 10^3\right] = \frac{\pi}{9} \times 999$
i.e. $V = 111\pi$ units^3 [3]

(c) (i) $x(1) = 3 + e^{-3} = 3{\cdot}049787\ldots$
So position when $t = 1$ is 3·05 cm (3 sig-figs) [1]

(ii) $v = \dot{x} = 3 - 3e^{-3t}$ cm/s
$\Longrightarrow v(0) = 3 - 3 = 0$ cm/s. i.e. at rest. [2]

(iii) $a = \dot{v} = 9e^{-3t}$ cm/s^2 [1]

(iv) As $t \to \infty$, $3e^{-3t} \to 0$
$\Longrightarrow$ the limiting velocity is 3 cm/s. [1]

QUESTION SEVEN

(a) (i) In $\triangle$'s ABC and AXY

(α) $\dfrac{AC}{AY} = \dfrac{AB}{AX} \left(= \dfrac{5}{4}\right)$

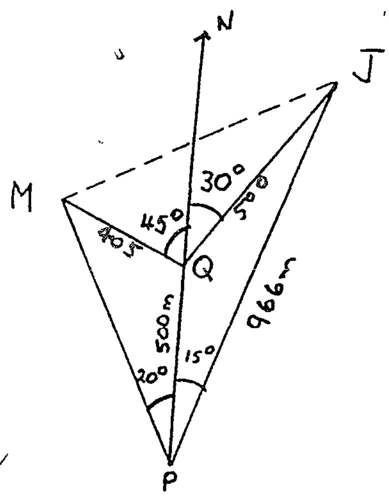
(β) $\angle A$ is common

$\implies \triangle ABC \;|||\; \triangle AXY$ (SAS) [2]

(ii) $\angle AXY = \angle ABC$ (matching $\angle$'s, similar $\triangle$'s equal)

$\implies XY \parallel BC$ as the corresponding angles are equal. [2]

(b) (i)



[1]

(i) $\angle PMQ = 25°$ (Exterior angle = sum interior opposite angles)

In $\triangle MQP$, $\dfrac{PM}{\sin 135°} = \dfrac{500}{\sin 25°}$

$\implies PM = \dfrac{500 \sin 135°}{\sin 25°}$

$\implies PM = 836{\cdot}578\ldots$

i.e. $PM = 837$ m to the nearest metre. [3]

(ii) In $\triangle MJP$, $MJ^2 = MP^2 + PJ^2 - 2MP.PL \cos 35°$

$\implies MJ^2 = 837^2 + 966^2 - 2 \times 837 \times 966 \cos 35°$

$\implies MJ = 555{\cdot}956\ldots$

i.e. $MJ = 556$ m to the nearest metre. [2]

(c) LHS $= \operatorname{cosec}^2\theta - \cot^2\theta$

$= \dfrac{1}{\sin^2\theta} - \dfrac{\cos^2\theta}{\sin^2\theta}$

$= \dfrac{1 - \cos^2\theta}{\sin\theta}$

$= \dfrac{\sin^2\theta}{\sin^2\theta}$

$= 1 =$ RHS [2]

QUESTION EIGHT

(a) (i) $m = Ae^{kt}$

$\Longrightarrow \dfrac{dm}{dt} = Ake^{kt}$

$\Longrightarrow \dfrac{dm}{dt} = km$ as required. [1]

(ii) (α) When $t = 0$, $m = 700$

So $700 = Ae^0 = A$ [1]

(β) When $t = 11$, $m = 600$

So $600 = 700e^{11k}$

$\Longrightarrow \frac{6}{7} = e^{11k}$

$\Longrightarrow 11k = \ln \frac{6}{7}$

i.e. $k = \frac{1}{11} \ln \frac{6}{7}$, as required. [1]

(γ) When $t = 100$: $m = 700e^{\frac{1}{11} \ln \frac{6}{7} \times 100}$

i.e. $m = 172{\cdot}381\ldots$

$\Longrightarrow m = 172$ g to the nearest gram. [1]

(δ) When $m = 100$ $100 = 700e^{\frac{1}{11} \ln \frac{6}{7} t}$

$\Longrightarrow \ln \frac{1}{7} = \frac{1}{11} \ln \frac{6}{7} t$

$\Longrightarrow t = \dfrac{11 \ln \frac{1}{7}}{\ln \frac{6}{7}}$

$\Longrightarrow t = 138{\cdot}8577\ldots$

Therefore there will be less than 100g after 139 weeks. [1]

(b) (i) 40°, 36°, $32{\cdot}4^\circ$ and $29{\cdot}16^\circ$ [1]

(ii) Geometric as $\frac{36}{40} = \frac{32{\cdot}4}{36} = 0{\cdot}9$ or $36^2 = 40 \times 32{\cdot}4$ [1]

(iii) $u_n = 40(0{\cdot}9)^{n-1}$ [1]

(iv) Require n such that $u_n < 1$

i.e. $40(0{\cdot}9)^{n-1} < 1$

$\Longrightarrow (0{\cdot}9)^{n-1} < \frac{1}{40}$

$n - 1 > \dfrac{\ln \frac{1}{40}}{\ln 0{\cdot}9}$

i.e. $n > 1 + \dfrac{\ln \frac{1}{40}}{\ln 0{\cdot}9} \doteqdot 36{\cdot}01197\ldots$

So the first swing through less than 1° is the 37 th swing. [2]

(v) $S_{10} = \dfrac{40(1 - 0{\cdot}9^{10})}{1 - 0{\cdot}9}$

$\Longrightarrow S_{10} = 260{\cdot}5286\ldots$

i.e. the pendulum has swung through a total of 261°

(to the nearest degree). [1]

(c) The limiting sum $= \dfrac{40^\circ}{1-0{\cdot}9} = 400^\circ$.
This means that the total angle the pendulum swings through is always less than 400°. [1]

QUESTION NINE

(a) (i) $A = \frac{1}{2} \times 12^2 \sin 60^\circ$
$\Longrightarrow A = 72 \times \frac{\sqrt{3}}{2}$
So $A = 36\sqrt{3}\ \text{cm}^2$ [1]

(ii) Area Sector $ARQ = \frac{1}{2} \times 6^2 \times \frac{\pi}{3}$
i.e. Area $= 6\pi\ \text{cm}^2$ [1]

(iii) Shaded area $= 36\sqrt{3} - 3 \times 6\pi$
$\Longrightarrow$ area $= 18(2\sqrt{3} - \pi)\ \text{cm}^2 = 5{\cdot}80516\ldots$
So the shaded area $= 5{\cdot}81\ \text{cm}^2$(3 sig. figs) [1]

(b) (i) At B, $x = 0$ so B is $(0, 10)$ [1]

(ii) $f'(x) = 4x^3 - 16x$ [1]

(iii) $f'(x) = 0 \Longrightarrow 4x(x^2 - 4) = 0$
$\Longrightarrow x = 0$ or $x^2 = 4$
i.e. $x = 0$ or ± 2 [1]

(iv) Hence A is $(-2, -6)$ and C is $(2, 6)$
Now $f''(x) = 12x^2 - 16$
$\Longrightarrow f''(\pm 2) = 32 \Longrightarrow A$ and C are minima.
$f''(0) = -16 \Longrightarrow$ B is a maximum. [3]

(c) $2\cos^2(\frac{\pi}{3} - x) = 1$
$\Longrightarrow \cos^2(\frac{\pi}{3} - x) = \frac{1}{2}$
$\Longrightarrow \cos(\frac{\pi}{3} - x) = \pm\frac{1}{\sqrt{2}}$
$\Longrightarrow \cos(x - \frac{\pi}{3}) = \pm\frac{1}{\sqrt{2}}$
So $x - \frac{\pi}{3} = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$
i.e. $x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$, or $\frac{19\pi}{12}$. [3]

QUESTION TEN

(a) (i) $A =$ rate of consumption$\times$ time.

i.e. $A = (50 + 0{\cdot}002v^3) \times \dfrac{\text{distance}}{\text{speed}}$

$\Longrightarrow A = (50 + 0{\cdot}0002v^3) \times \dfrac{9000}{v}$

$\Longrightarrow A = \dfrac{450\ 000}{v} + 18v^2$ as required. [1]

(ii) $\dfrac{dA}{dv} = \dfrac{-450\ 000}{v^2} + 36v$

$\dfrac{dA}{dv} = 0 \Longrightarrow 36v^3 = 450\ 000$

$\Longrightarrow v^3 = 12\ 500$

$\Longrightarrow v = 23{\cdot}2079\ldots$

So $v = 23{\cdot}2$ km/hr (3 sig-figs)

Now $\dfrac{d^2A}{dv^2} = \dfrac{900\ 000}{v^3} + 36$

Clearly when $v = 23{\cdot}2$ or $v^3 = 12\ 500$, $\dfrac{d^2A}{dv^2} = 108 > 0$

$\Longrightarrow A$ is a minimum when $v = 23{\cdot}2$.

When $v = 23{\cdot}2$, $A = \dfrac{450\ 000}{23{\cdot}2} + 18 \times 23{\cdot}2^2 = 29\ 084{\cdot}86\ldots = 29\ 100$ (3 sig-figs)

Therefore the greatest fuel economy occurs when the ship travels at 23·2 km/hr and 29 100 tonnes of fuel is used. [3]

(b) (i) $f(x) = \dfrac{e^x - e^{-x}}{2}$

$\Longrightarrow f'(x) = \dfrac{e^x + e^x}{2}$

i.e. $f'(x) = g(x)$

$g(x) = \dfrac{e^x + e^{-x}}{2}$

$\Longrightarrow g'(x) = \dfrac{e^x - e^{-x}}{2}$

i.e. $g'(x) = f(x)$ [1]

(ii) Now $f'(x) = g(x) > 0$ as both e^x and $e^{-x} > 0$ for all real x.
i.e. $f(x)$ is increasing for all values of x.
Furthermore $f''(x) = g'(x) = f(x)$ and $f(-1) = \frac{1}{2}(\frac{1}{e} - e) < 0$; $f(0) = 0$; and $f(1) = \frac{1}{2}(e - \frac{1}{e}) > 0 \Longrightarrow$ there is a point of inflexion at the origin. [2]

(iii) $g'(x) = f(x) = 0 \Longrightarrow x = 0$
Also $g''(x) = f'(x) = g(x) \Longrightarrow g''(0) = \frac{1}{2} > 0$
$\Longrightarrow$ the point $(0, 1)$ is a minima. [1]